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TECHNOLOGY DRIVEN. WARFIGHTER FOCUSED.

# Math-Based Simulation Tools and Methods

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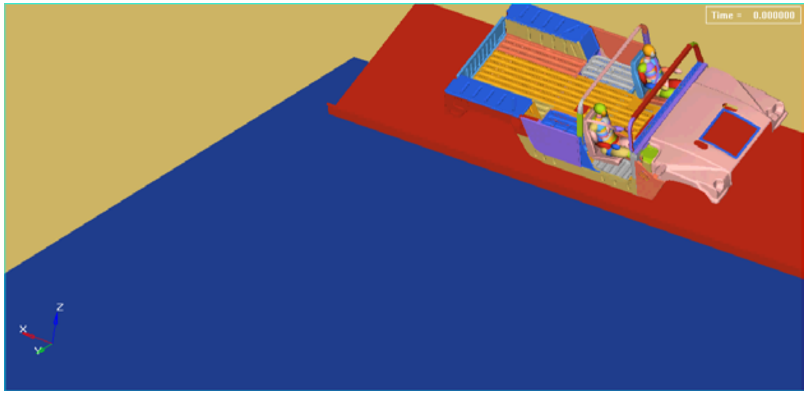
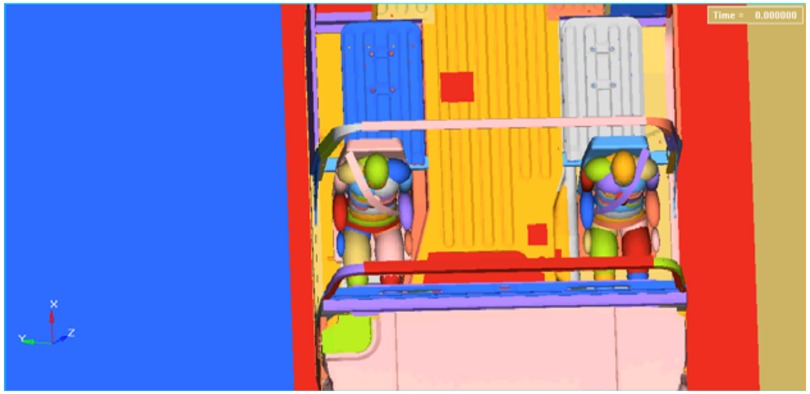
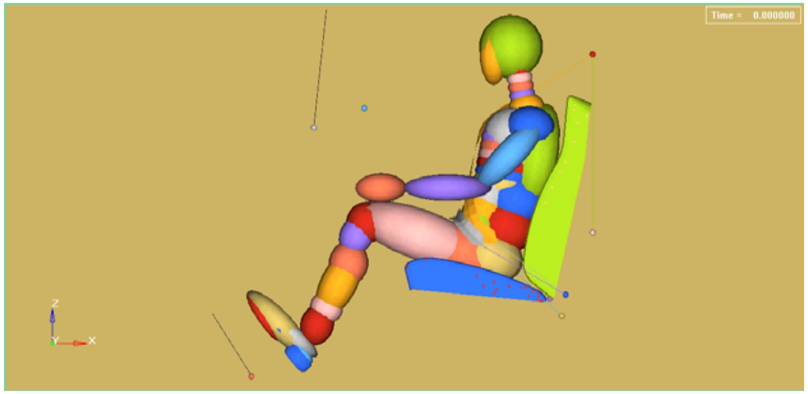
# HMMWV 30-mph Rollover Test

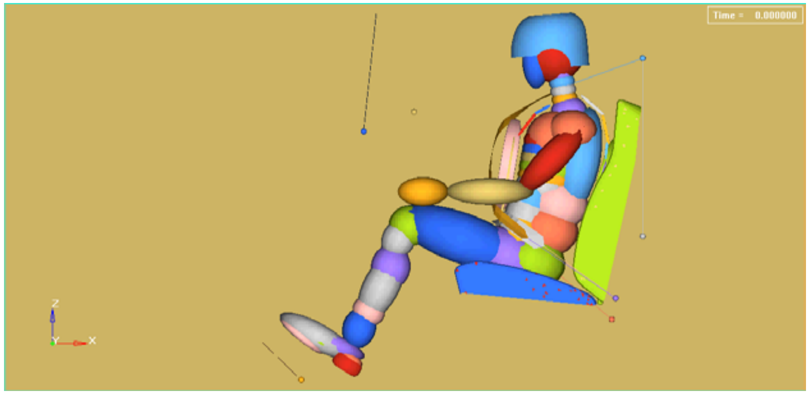


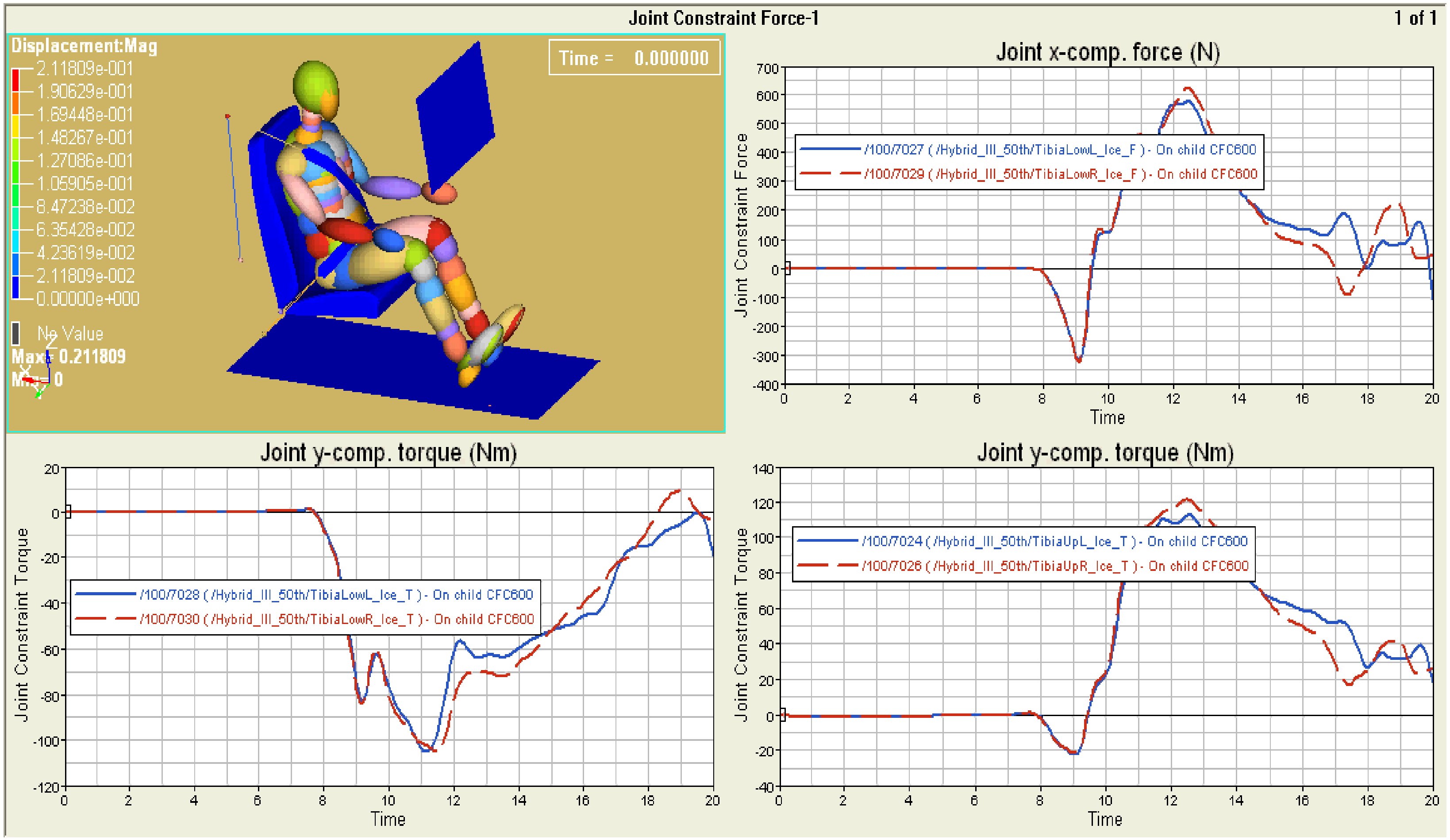
# Soldier Gear Effects







# Occupant Performance in Blast Effects



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# Anthropomorphic Test Device

TARDEC


Head

Neck

Chest

Abdomen

Lumbar

Femur

Knee

Tibia

Ankle

Foot

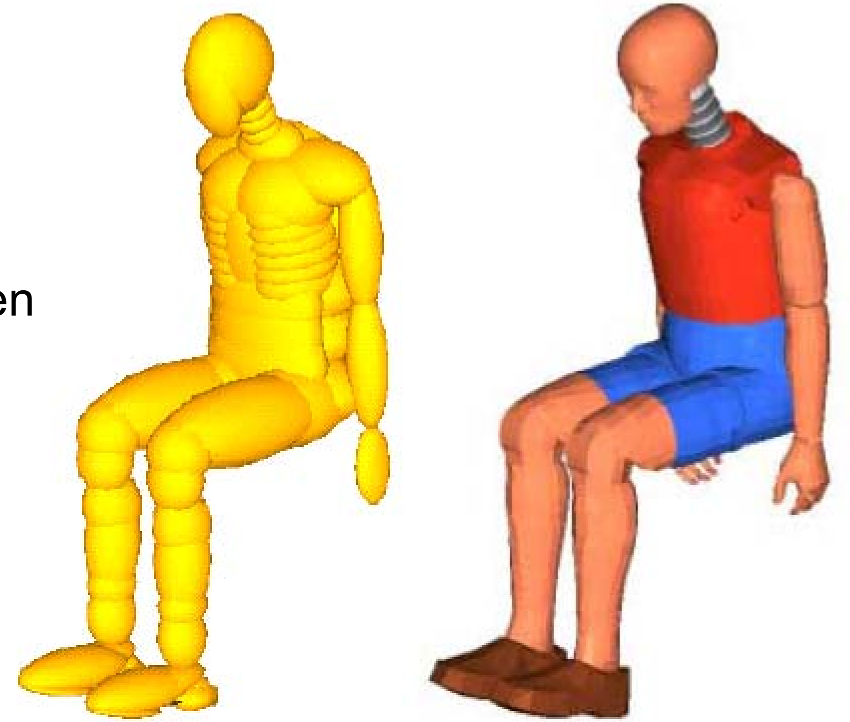
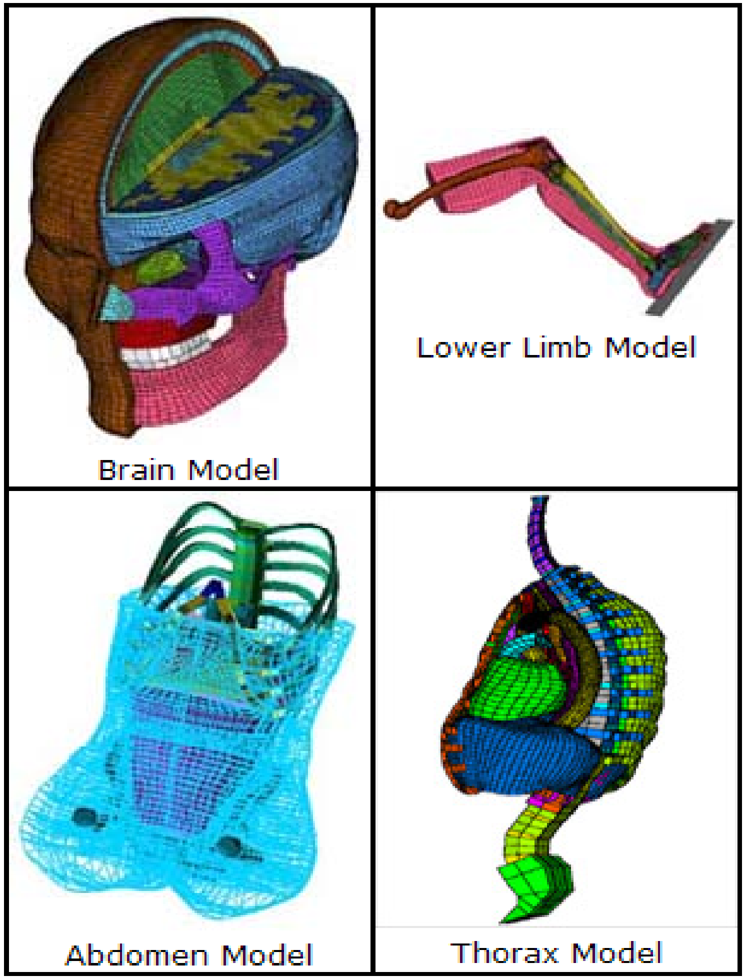


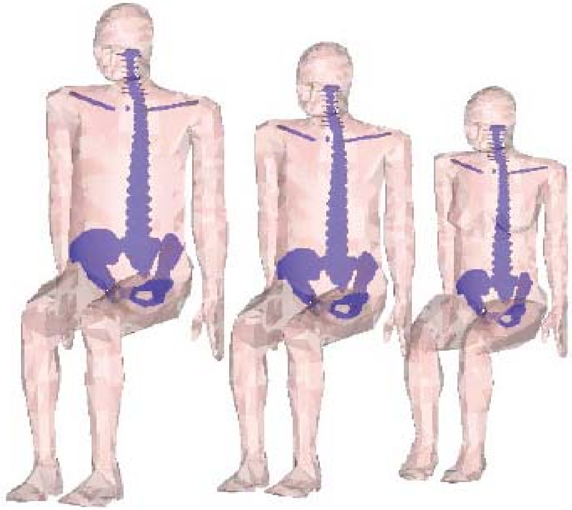
Figure 2.1 Hybrid III 50th percentile dummy model; ellipsoid model (left) and facet model (right).

# Human Models

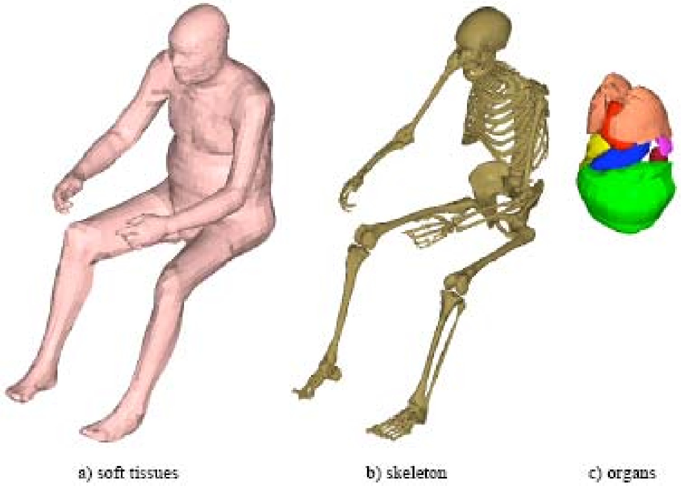



Wayne State - ANSIR

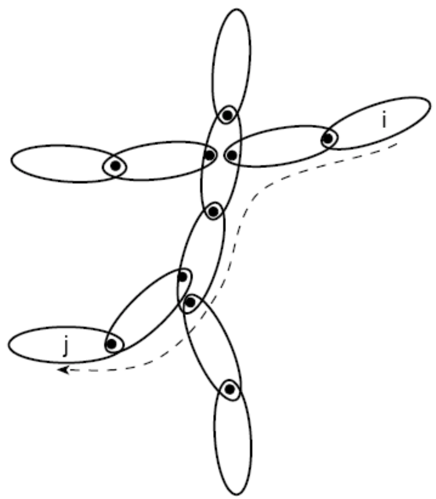
MADYMO- FACET



MADYMO FE- HUMOS2



# Rigid Body Modeling




*A multibody system is a set of bodies interconnected by a kinematic joints*

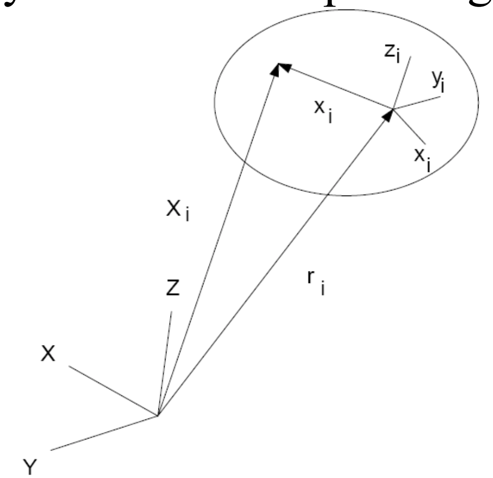
KINEMATICS OF A RIGID BODY

Position vector of P, a point on body $i$ in reference space is given by,

$$X_i = r_i + x_i$$

or in matrix notation,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_i = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}_i + \begin{pmatrix} A_{11} A_{12} A_{13} \\ A_{21} A_{22} A_{23} \\ A_{31} A_{32} A_{33} \end{pmatrix}_i \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_i$$




First time derivative equals,

$\dot{X}_i = \dot{r}_i + \omega_i \times x_i$, where $\omega_i$ is the angular velocity vector of body $i$

And 2nd time derivative equals,

$$\ddot{X}_i = \ddot{r}_i + \dot{\omega}_i \times x_i + \omega_i \times (\omega_i \times x_i)$$

# Rigid body modeling - continued

KINEMATICS OF BODIES CONNECTED BY JOINT

$$\underline{A}_j = \underline{A}_i \underline{C}_{ij} \underline{D}_{ij} \underline{C}^T{}_{ji} \qquad \Rightarrow 1$$

Where,

$\underline{C}_{ij}$ = time dependent direction cosine matrix, joint coordinate system on body $i$ relative to body local coordinate system on body $i$

$D_{ij}$ = direction cosine matrix, orientation of joint coordinate system on body $j$ with relative to joint coordinate system on body $i$

$A_i$ = direction cosine matrix, orientation of local coordinate system of body $i$

$A_j$ = direction cosine matrix, orientation of local coordinate system of body $j$

$$r_j = r_i + c_{ij} + d_{ij} - c_{ji} \qquad \Rightarrow 2$$

$r_j$ = position vector of the origin of the local coordinate system on body $j$

$r_i$ = position vector of the origin of the local coordinate system on body $i$

$c_{ij}$ = position vector of the origin of the joint coordinate system on body $i$ relative to the origin of the local coordinate system of the corresponding body

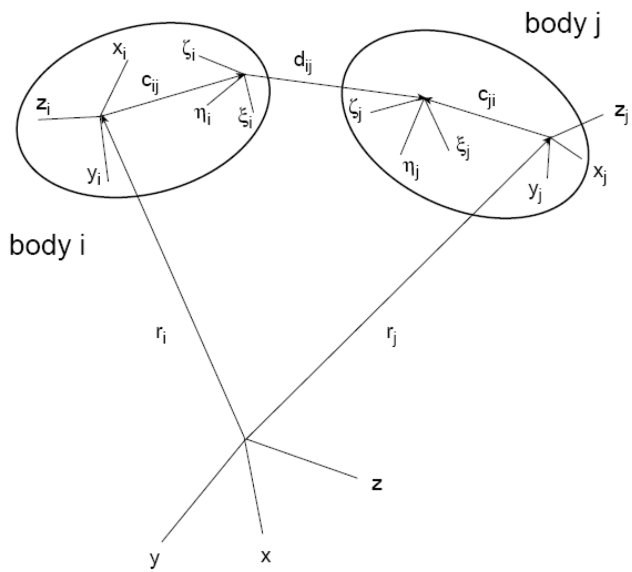
$c_{ji}$ = position vector of the origin of the joint coordinate system on the body $j$ relative to the origin of the local coordinate system of the corresponding body

$d_{ij}$ = vector from the origin of the joint coordinate system on body $i$ to the joint coordinate system on body $j$

Taking first time derivatives, yield the following expressions for angular and linear velocities -

$$\omega_j = \omega_i + \omega_{ij} \qquad \Rightarrow 3$$

$$\dot{r}_j = \dot{r}_i + \omega_i \times c_{ij} + \dot{d}_{ij} - \omega_j \times c_{ji} \qquad \Rightarrow 4$$




*Two bodies interconnected by an arbitrary kinematic joint; motion of body j is described relative to parent body i*

# Rigid body modeling - continued



Equations of motion (Newton - Euler) of rigid body $i$ referred to its center of gravity are:

$$m_i \ddot{r}_i = F_i$$

$$J_i \cdot \dot{\omega}_i + \omega_i \times J_i \cdot \omega_i = T_i$$

Where,

$m_i$ = mass

$J_i$ = inertia tensor w/r to center of gravity

$\omega_{\mathrm{i}}$ = angular velocity vector

$F_i$ = resultant force vector relative to the CG

$T_i$ = resultant torque vector relative to the CG

$F_i$ and $T_i$ both include constraint forces and torques due to joints

Using principle of virtual work and variation of position vector, $\delta r_i$ and a variation of orientation, $\delta \pi_i$, the resulting equations summed for all bodies in the system:

$$\sum \delta r_i \cdot \{m_i \ddot{r}_i - F_i\} + \delta \pi_i \cdot \{J_i \cdot \dot{\omega}_i + \omega_i \times J_i \cdot \omega_i - T_i\} = 0$$

# Rigid body modeling - continued



Expression for 2nd derivative of the joint degrees of freedom are obtained :

$$\underline{\ddot{q}}_{ij} = \underline{M}_{ij}\underline{\dot{Y}}_i + \underline{Q}_{ij}$$

Where,

$\underline{\dot{Y}}_i$ is a 6 x 1 column matrix that contains components of linear and angular acceleration of the coordinate system of the parent body $i$

$n_{ij}$ x 6 matrix $\underline{M}_{ij}$ and the $n_{ij}$ x 1 column matrix of $\underline{Q}_{ij}$ depend on the inertia of the bodies and the instantaneous geometry of the system. $\underline{Q}_{ij}$ depends additionally on the instantaneous geometry of the system and the applied loads.

# Rigid body modeling - continued

Numerical integration methods:

The equations of motion form a system of coupled non-linear second order differential equations:

$$\underline{\ddot{q}} = \underline{h}\left(\underline{q}, \underline{\dot{q}}, t\right), \text{ with initial values of } q_0 \text{ and } \dot{q}_0$$

$\underline{q}$ = column matrix with generalized coordinates; joint position degrees of freedom

$\underline{\dot{q}}$ = joint velocity degrees of freedom

$\underline{\ddot{q}}$ = joint acceleration degrees of freedom

Equations of motion are solved numerically using the available three methods:

- a modified Euler method with a fixed or variable time step
- a Runge-Kutta method with a fixed or variable time step
- a Runge-Kutta Merson method with a variable time step

Modified Euler method:

One step method with fixed time step $t_s$; second order differential equations are integrated explicit Euler method which gives velocity variables at time point $t_{n+1} = t_n + t_s$,

$$\underline{\dot{q}}_{n+1} = \underline{\dot{q}}_n + t_s \underline{\ddot{q}}_n$$

The velocity variables are integrated using the implicit Euler method which gives solution for the position variables at time point $t_{n+1}$,

$$\underline{q}_{n+1} = \underline{q}_n + t_s \underline{\dot{q}}_{n+1}$$

# Finite Element Methods

Displacement Matrix

$$\{\delta\}_e = \{u_i, v_i, u_j, v_j, u_m, v_m\}$$

or, displacement function at any point within the element is

$$\{f\}_e = [N]\{\delta\}_e$$

Stress, Strain and Elasticity Matrices

$$\{\sigma\}_e = [D]\{\varepsilon\}_e$$

Formulation of Finite Element Methodology using the principle (minimization) of total potential energy of the system -

$$\sum_1^n \int_V \left(\{\Delta\varepsilon\}_e^T \{\sigma\}_e - \{\Delta f\}_e^T \{F\}_e\right) dV - \sum_1^n \int_s \{\Delta f\}_e^T \{T\}_e \, ds = 0$$

$$\text{or, } \sum_1^n \{\Delta\delta\}_e^T \left([k]_e \{\delta\}_e - \{Q\}_e\right),$$

the stiffness matrix, $[k]_e$, and nodal force matrix, $\{Q\}_e$, (due to body force, initial strain, and surface traction) are

$$[k]_e = \int_V [B]^T [D][B] dV \text{ and}$$

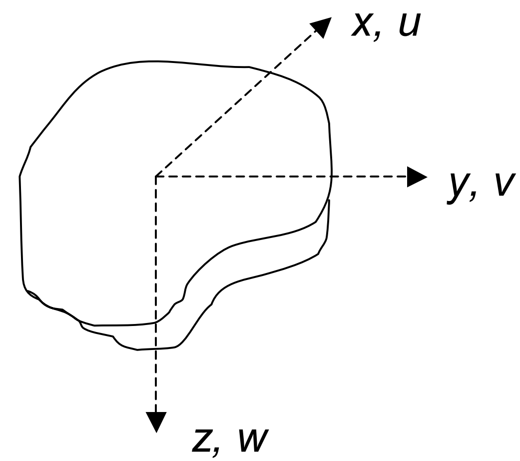
$$\{Q\}_e = \int_V [N]^T \{F\} dV + \int_V [B]^T [D]\{\varepsilon_0\} dV + \int_s [N]^T \{T\} ds$$

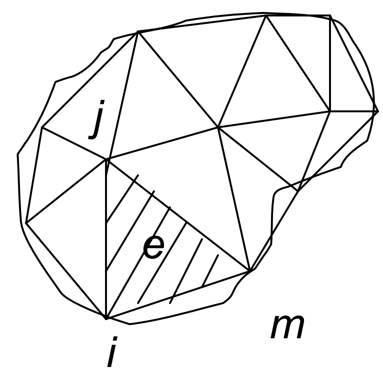
And the system of equations for equilibrium for nodal forces for the entire structure,

$$[K]\{\delta\} = \{Q\}$$

where,

$$[K] = \sum_1^n [k]_e, \text{ and } \{Q\} = \sum_1^n \{Q\}_e$$

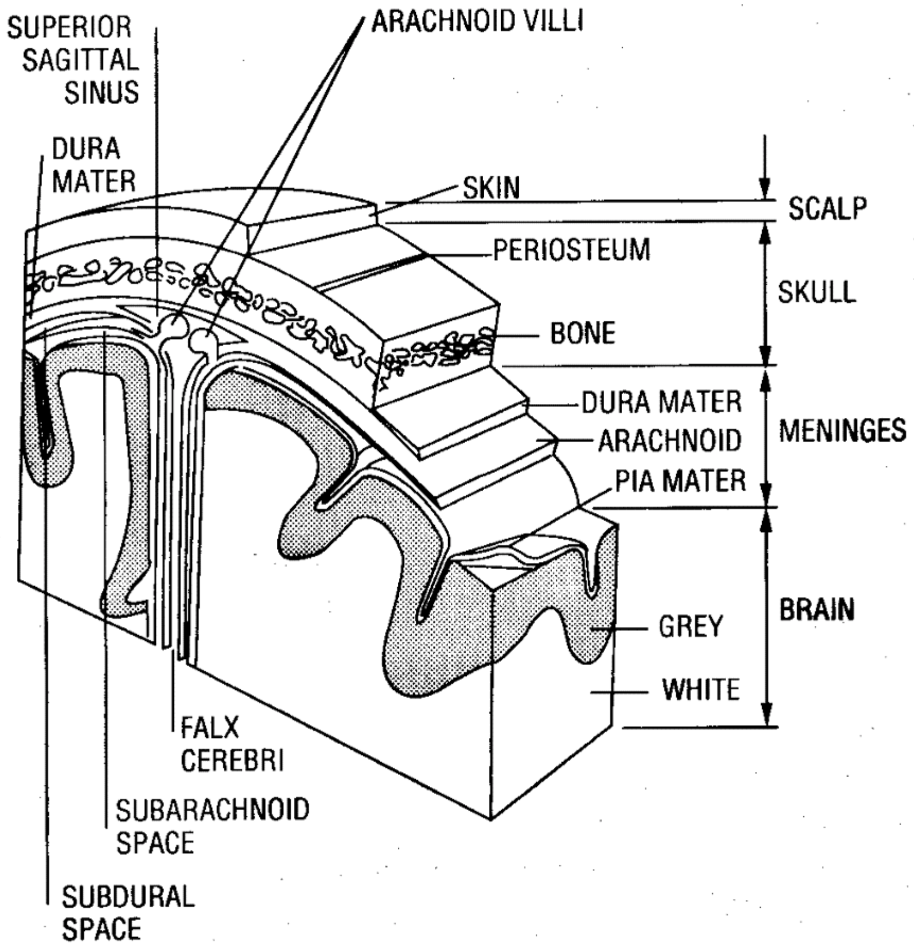



*Typical finite element 'e'*

# Injury Criteria development

### Scalp, Skull, Meninges and Brain



# Injury Criteria development - continued

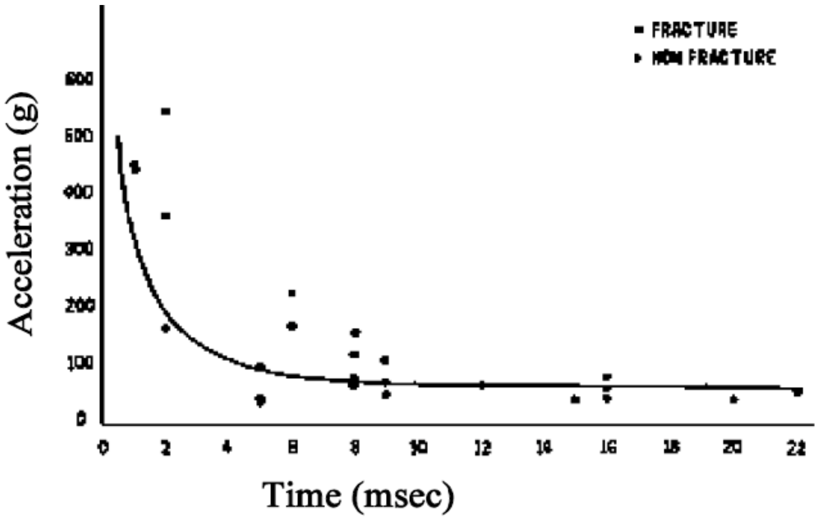
Head Injury Criteria :

$$HIC = \max\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)dt\right]^{2.5}(t_2 - t_1)$$

Where, $t_1$ and $t_2$ are two arbitrary times during acceleration pulse

HIC is limited to 36 ms

**Wayne State Tolerance Curve**




Lognormal distribution (pdf) :

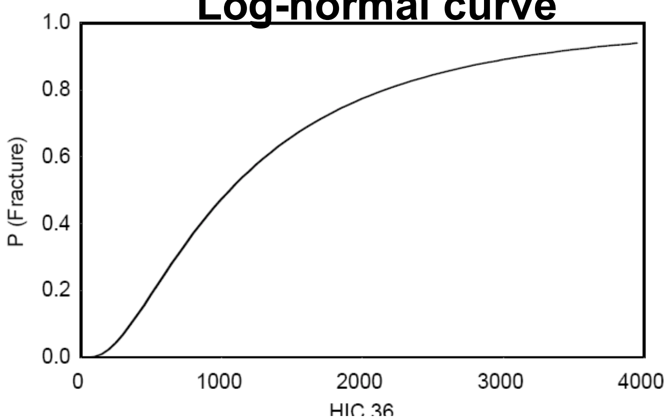
$$f(T') = \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T'-\mu'}{\sigma_{T'}}\right)^2}$$

The probability of skull fracture (AIS≥2) is given by the formula

$$p\ (fracture) = N\left(\frac{\ln(HIC) - \mu}{\sigma}\right),$$

where N( ) is the cumulative normal distribution, μ = 6.96352 and σ = 0.84664.

**Injury Risk Curve for HIC Log-normal curve**



# Optimization



Non-linear Constrained Optimization Problem:

Minimize: $F(x)$ Objective function

Subject to:

$g_j(\mathrm{X}) \le 0 \qquad j = 1, m$ Inequality constraint

$h_k(\mathrm{X}) = 0 \qquad k = 1, l$ Equality constraint

$X_i^l \le X_i \le X_i^u \qquad i = 1, n$ Side constraints

$$\text{where, } \mathrm{X} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ . \\ . \\ . \\ X_n \end{Bmatrix}$$

Necessary and sufficient conditions for global optimum for unconstrained problems -

$\nabla \mathrm{F}(\mathrm{X}) = 0$ (the gradient must vanish)

# Optimization - continued



where,

$$\nabla \mathrm{F(X)} = \begin{Bmatrix} \frac{\partial}{\partial X_1} F(X) \\ \frac{\partial}{\partial X_2} F(X) \\ . \\ . \\ . \\ \frac{\partial}{\partial X_n} F(X) \end{Bmatrix}$$

and the Hessian matrix must be positive definite

$$\mathrm{H} = \begin{bmatrix} \frac{\partial^2 F(X)}{\partial X_1^2} & \frac{\partial^2 F(X)}{\partial X_1 \partial X_2} \cdots & \frac{\partial^2 F(X)}{\partial X_1 \partial X_2} \\ \frac{\partial^2 F}{\partial X_2 \partial X_1} & \frac{\partial^2 F(X)}{\partial X_2^2} \cdots & \frac{\partial^2 F(X)}{\partial X_2 \partial X_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 F(X)}{\partial X_n \partial X_1} & \frac{\partial^2 F(X)}{\partial X_n \partial X_2} \cdots & \frac{\partial^2 F(X)}{\partial X_n^2} \end{bmatrix}$$

And for constrained problems, if vector $\mathrm{X}^*$ defines the optimum design,
Kuhn - Tucker conditions are necessary and sufficient :

1. $X^*$ is feasible

2. $\lambda_j g_j\left(X^*\right) = 0 \qquad j = 1, m \qquad \lambda_j \geq 0$

3. $\nabla F\left(X^*\right) + \sum_{j=1}^{m} \lambda_j \nabla g_j\left(X^*\right) + \sum_{k=1}^{l} \lambda_{m+k} \nabla h_k\left(X^*\right)$

$\lambda_j \geq 0$

# Summary

- Crashworthiness and occupant protection in motor vehicles comprise the following significant mathematics applications –
  - Matrix operations
  - Ordinary and partial differential system of equations
  - Lagrangian operations
  - Fourier transforms
  - Taylor series
  - Finite difference methods
  - Implicit and explicit finite element methods
  - Statistical methods – probabilistic and regression analysis